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Characterizations of Real Hypersurfaces in a Complex Space Form in Terms of Lie Derivatives*

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Abstract

In this paper we study real hypersurfaces M in a non-flat complex space form by using the Lie derivatives on M.

Keywords and Phrases: *complex space form, Hopf hypersurfaces, Lie derivatives.*

1. Introduction

Let $M_n(c)$ be an n-dimensional complete and simply connected non-flat complex space form with constant holomorphic sectional curvature $4c$. Then $M_n(c)$ is either a complex projective space CP^n (for $c > 0$) or a complex hyperbolic space CH^n (for $c < 0$). For convenience, we assume that $c = 1$ for CP^n, and $c = -1$ for CH^n. Let M be a real hypersurface in $M_n(c)$. Then M has an almost contact metric structure $(\phi, \xi, \eta, \langle,\rangle)$ induced from the complex structure J of $M_n(c)$. Typical

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examples of real hypersurfaces are the six model spaces of type A_1, A_2, *B*, *C*, *D* and *E* in CP^n, and the four model spaces of type A_0, A_1, A_2 and *B* in CH^n.

One of the most interesting problems in the study of real hypersurfaces *M* in $M_n(c)$ is to obtain characterizations of these model spaces. In particular, many differential geometers characterizes *M* by using Lie derivatives on *M* (cf. [5], [9]). The purpose of this paper is to continue the study of real hypersurface *M* in $M_n(c)$ along this line.

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2. Preliminaries

Let *M* be an orientable connected real hypersurface in $M_n(c)$, $n \geq 3$, and let *N* be a unit normal vector field on *M*. Denote by $\tilde{\nabla}$ and ∇ respectively the Levi-Civita connection on $M_n(c)$ and that induced on *M*. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N$$

$$\tilde{\nabla}_X N = -AX$$

for any vector field *X* and *Y* tangent to *M*, where $\langle , \rangle$ denotes the Riemannian metric of *M* induced from the Riemannian metric of $M_n(c)$. Now, we define a tensor field ϕ of type (2,1), a vector field ξ and a 1-form η by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi$$

where ϕX is the tangential part of *JX*. Then it is seen that $\langle \xi, X\rangle = \eta(X)$. Furthermore, the set of tensors $(\phi, \xi, \eta, \langle , \rangle)$ is an almost contact metric structure on *M*, i.e., they satisfy the following

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

The covariant derivatives of the structure tensor are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle \xi, \quad \nabla_X \xi = \phi AX. \tag{2.2}$$

A real hypersurface M is said to be *totally η-umbilical* if

$$AX = aX + b\eta(X)\xi$$

for any vector field X tangent to M, where a and b are some functions on M. An eigenvalue of the shape operator A of M is called a *principal curvature* and a *principal curvature vector* is an eigenvector of A. A real hypersurface M in $M_n(c)$ is called a *Hopf hypersurface* if the structure tensor field ξ is principal. The *holomorphic distribution D* on M is defined as follows

$$D_x = \{X \in T_xM \mid \eta(X) = 0\}; \qquad x \in M.$$

In what follows, we say that a real hypersurface M is of type A when it is one of type A_0, A_1 or A_2; and we denote by $\Gamma(\mathbf{V})$ the module of all differentiable sections on the vector bundle $\mathbf{V}$ over M. Next, we recall some known results for later use.

Theorem 2.1. [4] *Let M be a real hypersurface in CP^n. Then M is a Hopf hypersurface with constant principal curvatures if and only if it is locally congruent to a tube of radius r over one of the following spaces:*

(A_1) *hyperplane CP^{n-1}, where $0 < r < \pi/2$,*

(A_2) *totally geodesic CP^p $(1 \le p \le n-2)$, where $0 < r < \pi/2$,*

(B) *complex quadric Q_{n-1}, where $0 < r < \pi/4$,*

(C) *$CP^1 \times CP^{(n-1)/2}$, where $0 < r < \pi/4$ and n (≥ 5) is odd,*

(D) *complex Grassmann $G_{2,5}$, where $0 < r < \pi/4$ and $n = 9$,*

(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.*

Theorem 2.2. [1] *Let M be a real hypersurface in CH^n. Then M is a Hopf hypersurface with constant principal curvatures if and only if it is locally congruent to one of the following spaces:*

(A_0) *a horosphere;*

(A_1) *a geodesic hypersphere CH^0 or a tube over a hyperplane CH^{n-1};*

(A_2) *a tube over a totally geodesic CH^p $(0 \le p \le n-2)$;*

(B) *a tube over a totally real hyperbolic space RH^n.*

Theorem 2.3. [8, 10] *Let M be a real hypersurface of $M_n(c)$. Then M is locally congruent to one of real hypersurfaces of type A if and only if $\phi A = A\phi$.*

Theorem 2.4. [7, 11] *Let M be a real hypersurface in* $M_n(c)$. *Then M is totally* η-*umbilical if and only if it is locally congruent to one of real hypersurfaces of type* A_0 *and* A_1.

Proposition 2.5. [2] *Let M be a real hypersurface in* $M_n(c)$. *If the structure vector* ξ *is principal then the corresponding principal curvature* α *is constant.*

Now, suppose M is a Hopf hypersurface in $M_n(c)$ and $A\xi = \alpha\xi$. Then we have [3]

$$2A\phi A = 2c\phi + \alpha(\phi A + A\phi). \tag{2.3}$$

$$\nabla_\xi A = \frac{\alpha}{2}(\phi A - A\phi). \tag{2.4}$$

Now, let $X \in \Gamma(D)$ be a principal curvature vector corresponding to the principal curvature λ, namely, $AX = \lambda X$. Then it follows from (2.3) that

$$(2\lambda - \alpha)A\phi X = (\alpha\lambda + 2c)\phi X. \tag{2.5}$$

Let us consider the open set

$$\mathbf{G} = \{x \in M \mid (2\lambda - \alpha)(x) \neq 0\}.$$

It turns out that on such a set **G**, ϕX is also a principal vector, i.e.,

$$A\phi X = \bar{\lambda}\phi X, \quad \bar{\lambda} = \frac{\alpha\lambda + 2c}{2\lambda - \alpha}. \tag{2.6}$$

Note that in the case $c > 0$, the set $\mathbf{G} = M$. Next, suppose that $c = -1$ ($c < 0$) and that the principal curvature λ is constant on the subset **G**. For each point $x \notin \mathbf{G}$, (1.5) implies that $\alpha\lambda - 2 = 0$. Hence $\alpha = \pm 2$ and $\lambda = \pm 1$. Since M is connected, **G** is either an empty set or coincides with M. If **G** is empty, then $\lambda = \pm 1$ and since M is connected, λ is a constant defined on the whole of M, and of course, λ is constant when $\mathbf{G} = M$.

Remark 2.1. By using a similar argument as above, if λ takes only finitely many values on the open set **G** then it is a constant on M.

3. Characterization of Real Hypersurfaces of Type *B*

In [5], Kimura and Maeda obtained a characterization of real hypersurfaces of type B in CP^n, namely, they showed the following:

Proposition 3.1. *Let M be a real hypersurface in* CP^n. *Then the following are equivalent:*

(1) ξ *is principal and* $(L_\xi\phi)^2 = -k^2\phi^2$, *where k is a non zero constant;*

(2) M is locally congruent to a real hypersurface of type B.

In fact, we shall show in the next proposition that the condition "ξ is a principal vector" of Proposition 3.1 can be dropped, and that this result is also true when the ambient space is CH^n. Namely, we prove the following:

Proposition 3.2. *Let M be a real hypersurface in* $M_n(c)$. *Then the following are equivalent:*

(1) $(L_\xi\phi)^2 = -k^2\phi^2$, *where k is a non zero constant;*

(2) M is locally congruent to a real hypersurface of type B.

Proof. For any $X \in \Gamma(TM)$, by using (2.1) and (2.2) we have

$$\begin{aligned}(L_\xi\phi)X &= [\xi, \phi X] - \phi[\xi, X]\\ &= (\nabla_\xi\phi)X - \nabla_{\phi X}\xi + \phi\nabla_X\xi\\ &= \eta(X)A\xi - \phi A\phi X - AX\,.\end{aligned}$$

Therefore

$$\begin{aligned}(L_\xi\phi)^2X &= \eta(X)\eta(A\xi)A\xi - \eta(AX)A\xi - \eta(X)A^2\xi + A^2X + A\phi A\phi X\\ &\quad - \eta(X)\phi A\phi A\xi + \phi A\phi AX + \phi A\phi^2A\phi X\\ &= \eta(X)\{\eta(A\xi)A\xi - A^2\xi - \phi A\phi A\xi\} + \eta(A\phi X)\phi A\xi + T^2X\end{aligned}$$

where $T = \phi A - A\phi$. From this equation, we see that $(L_\xi\phi)^2 = -k^2\phi^2$ is equivalent to

$$T^2X = \langle\phi A\xi, X\rangle\phi A\xi + \eta(X)H - k^2\phi^2X \qquad (3.1)$$

where $H = \phi A\phi A\xi + A^2\xi - \eta(A\xi)A\xi$. By taking inner product with $Y \in \Gamma(TM)$ we get

$$\langle T^2X, Y\rangle = \langle \phi A\xi, X\rangle\langle \phi A\xi, Y\rangle + \eta(X)\langle H, Y\rangle - k^2\langle \phi^2X, Y\rangle .$$

Together with the fact that T^2 is symmetric, we obtain

$$\eta(Y)\langle H, X\rangle = \eta(X)\langle H, Y\rangle .$$

By putting $Y = \xi$ in the above equation we see that

$$\langle H, X\rangle = \eta(H)\langle \xi, X\rangle , \qquad \text{for any } X \in \Gamma(TM)$$

which implies that H is proportional to ξ. Thus,

$$H = \eta(\phi A\phi A\xi + A^2\xi - \eta(A\xi)A\xi)\xi = \|\phi A\xi\|^2\xi = \|V\|^2\xi$$

where $V = \phi A\xi$. Therefore, (3.1) becomes

$$T^2X = \langle V, X\rangle V + \|V\|^2\eta(X)\xi - k^2\phi^2X . \tag{3.2}$$

By putting $X = \xi$ in (3.2) we obtain

$$T^2\xi = \|V\|^2\xi . \tag{3.3}$$

On the other hand, we have

$$T^2\xi = T(\phi A - A\phi)\xi = TV .$$

From which, together with (3.3), gives

$$T^2V = \|V\|^2 T\xi = \|V\|^2 V .$$

Now, if we put $X = V$ in (3.2) then

$$T^2V = \|V\|^2 V + k^2V .$$

These equations show that $V = \phi A\xi = 0$ (since $k \neq 0$), which means that ξ is principal, say $A\xi = \alpha\xi$. If $c > 0$ then our statement follows from Proposition 3.1. Now suppose that $c = -1 < 0$. Since ξ is principal (or $V = 0$), (3.2) reduces to

$$T^2X = -k^2\phi^2X$$

namely,

$$\phi A\phi AX - A\phi^2AX - \phi A^2\phi X + A\phi A\phi X = -k^2\phi^2X .$$

Let $X \in \Gamma(D)$ be a principal curvature vector with principal curvature λ. On the open set **G**, by (2.6) we have

$$(\lambda - \bar{\lambda})^2 = k^2 \tag{3.4}$$

Therefore, λ takes only finitely many values on **G** and so it is constant on M. By virtue of Theorem 2.2, M is locally congruent to one of real hypersurfaces of type A and type B Moreover, (3.4) shows that $\lambda - \bar{\lambda} \neq 0$ or equivalently, $\phi A - A\phi \neq 0$, hence Theorem 2.3 implies that M is not of type A. Now we consider M of type B (which is a tube of radius $r > 0$). Then M has at most three distinct constant principal curvatures $\alpha = 2\tanh 2r$, $\lambda_1 = \tanh r$ and $\lambda_2 = \coth r$ (cf. [1]). Since $\phi V_{\lambda_1} = V_{\lambda_2}$, (3.4) gives

$$k^2 = (\tanh r - \coth r)^2 = \frac{4}{\sinh^2 2r}.$$

Therefore, we can choose an appropriate k for real hypersurface of type B.

4. Characterization of Real Hypersurfaces of Type *A*

Let M be a real hypersurface of type A in $M_n(c)$. Then it follows from Theorem 2.3 that, M naturally satisfies $L_\xi(\phi A - A\phi) = 0$. In this section, we study the converse problem. Namely, we shall give another characterization of real hypersurfaces of type A under the condition $L_\xi(\phi A - A\phi) = 0$.

Proposition 4.1. *Let M be a real hypersurface in $M_n(c)$ and let $T = \phi A - A\phi$. Then M is locally congruent to one of real hypersurfaces of type A if and only if $L_\xi T = 0$.*

Proof. For any $X \in \Gamma(TM)$, we have

$$(L_\xi T)X = [\xi, TX] - T[\xi, X] = (\nabla_\xi T)X - \phi ATX + T\phi AX.$$

It follows from (2.2) and the above equation that $L_\xi T = 0$ is equivalent to

$$\begin{aligned}\phi ATX - T\phi AX &= (\nabla_\xi \phi)AX + \phi(\nabla_\xi A)X - (\nabla_\xi A)\phi X - A(\nabla_\xi \phi)X \\ &= \phi(\nabla_\xi A)X - (\nabla_\xi A)\phi X + 2\eta(AX)A\xi - \eta(A^2X)\xi - \eta(X)A^2\xi.\end{aligned}$$

By taking the inner product with $Y \in \Gamma(TM)$, the above equation becomes

$$\begin{aligned}\langle \phi ATX, Y\rangle - \langle T\phi AX, Y\rangle = \langle \phi(\nabla_\xi A)X, Y\rangle - \langle (\nabla_\xi A)\phi X, Y\rangle + 2\eta(AX)\eta(AY) \\ - \eta(Y)\eta(A^2X) - \eta(X)\eta(A^2Y).\end{aligned} \tag{4.1}$$

On the other hand, we see that

$$\langle \phi ATX, Y\rangle - \langle T\phi AX, Y\rangle = \langle A\phi X, A\phi Y\rangle - \langle \phi AX, \phi AY\rangle .$$

Thus, (4.1) reduces to

$$\langle A\phi X, A\phi Y\rangle - \langle \phi AX, \phi AY\rangle = \langle \phi(\nabla_\xi A)X, Y\rangle - \langle (\nabla_\xi A)\phi X, Y\rangle + 2\eta(AX)\eta(AY) - \eta(Y)\eta(A^2X) - \eta(X)\eta(A^2Y) . \tag{4.2}$$

By putting $X = Y = \xi$ in (4.2), we obtain $\langle \phi A\xi, \phi A\xi\rangle = 0$. In other words, the structure vector field ξ is principal, say $A\xi = \alpha\xi$. Hence (4.2) can be reduced to

$$\langle A\phi X, A\phi Y\rangle - \langle \phi AX, \phi AY\rangle = \langle \phi(\nabla_\xi A)X, Y\rangle - \langle (\nabla_\xi A)\phi X, Y\rangle .$$

By letting $Y = X \in \Gamma(D)$ in the above equation and in view of (2.4), we get

$$\langle A\phi X, A\phi X\rangle - \langle \phi AX, \phi AX\rangle = \alpha\{\langle A\phi X, \phi X\rangle - \langle AX, X\rangle\} .$$

Now, let $X \in \Gamma(D)$ be a unit vector with $AX = \lambda X$ and let us restrict our discussion to the subset **G** at the moment. Then it follows from (2.6) that the above equation gives

$$(\bar\lambda - \lambda)(\bar\lambda + \lambda - \alpha) = 0 . \tag{4.3}$$

In the following, our discussion is divided into two cases (I) $c = 1$ ($M_n(c) = CP^n$) and (II) $c = -1$ ($M_n(c) = CH^n$).

Case (I): We first note that $\mathbf{G} = M$ in this case. Next, by (2.6) we can see that

$$\bar\lambda + \lambda - \alpha = \frac{\alpha\lambda + 2}{2\lambda - \alpha} + \lambda - \alpha = \frac{2\lambda^2 - 2\alpha\lambda + \alpha^2 + 2}{2\lambda - \alpha} \neq 0 .$$

Consequently, we must have $\bar\lambda = \lambda$, which is equivalent to $\phi A = A\phi$ and we conclude that M is locally congruent to one of type A according to Theorem 2.3.

Case (II): In this case, (4.3) implies that λ is constant on M. Hence, Theorem 2.2 asserts that M is locally congruent to either one of type A or type B. As an immediate consequence of Theorem 2.3, we see that the rest of the proof is to show that a type B real hypersurface M does not satisfy $L_\xi(\phi A - A\phi) = 0$.

Let M be locally congruent to one of type B. Then M has at most three distinct constant principal curvatures α, λ_1 and λ_2 (as in the Proof of Proposition 3.2). Now we set $\lambda = \tanh r$ $(= \lambda_1)$. Then $\bar\lambda = \coth r$ since $\phi V_{\lambda_1} = V_{\lambda_2}$. It follows that

$$\bar\lambda - \lambda = \coth r - \tanh r = \frac{2}{\sinh 2r} \neq 0 .$$

$$\bar{\lambda} + \lambda - \alpha = \coth r + \tanh r - 2\tanh 2r = 2(\coth 2r - \tanh 2r) \neq 0 .$$

Hence M does not satisfy (4.3) and this completes the proof.

5. Characterization of Model Spaces

We first prove the following:

Proposition 5.1. *Let M be a real hypersurface in $M_n(c)$. Then M is locally congruent to one of type A_0, A_1 and B if and only if*

$$\phi A + A\phi - k\phi = 0 \tag{5.1}$$

for some non zero constant k.

Proof. From (5.1) we obtain $\phi A\xi = 0$. Namely, ξ is principal (say $A\xi = \alpha\xi$). For any $X \in \Gamma(D)$ with $AX = \lambda X$ and at any point x in **G**, we have

$$\lambda + \bar{\lambda} - k = 0$$

or

$$\lambda^2 - k\lambda + c + \frac{\alpha k}{2} = 0 . \tag{5.2}$$

Therefore, λ is constant on M and at most two λ are distinct. So we conclude that M is locally congruent to either one of real hypersurfaces of A or B. Next we shall check for each case one by one:

Let M be locally congruent to one of type A. Then by Theorem 2.3 and (5.1) we get

$$2A\phi - k\phi = 0$$

and so $AX = \lambda X$ for all $X \in \Gamma(D)$. Hence, M is totally η-umbilical and we conclude that M is locally congruent to one of type A_0 or A_1 (by Theorem 2.4).

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Now, let M be locally congruent to one of type B. Then M has at most three distinct constant principal curvatures α, λ_1 and λ_2 (as in the Proof of Proposition 3.2), it is clear that λ_1 and λ_2 are the solutions of (5.2) for $k = -\dfrac{4c}{\alpha}$.

From the proof of Proposition 5.1, we have the following.

Corollary 5.2. *Let M be a real hypersurface in* $M_n(c)$. *If* $\alpha = \eta(A\xi)$ *is nowhere zero, then M is locally congruent to one of type B real hypersurfaces if and only if*

$$\phi A + A\phi + \frac{4c}{\alpha}\phi = 0 .$$

Remark 5.1. Proposition 5.1 was proved by Kon [6] for $c > 0$.

Now, we prove the main result in this section.

Theorem 5.3. *Let M be a real hypersurface of* $M_n(c)$ *and let* $Q = \phi A + A\phi - k\phi$, *where k is a nonzero constant. If M satisfies* $L_\xi Q = 0$ *then it is locally congruent to one of the model spaces A, B, C, D and E.*

Proof. For any $X \in \Gamma(D)$, we have

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$$(L_\xi Q)X = [\xi, QX] - Q[\xi, X] = (\nabla_\xi Q)X - \phi AQX + Q\phi AX .$$

On the other hand, we see that

$$\phi AQ - Q\phi A = \phi A^2\phi - A\phi^2 A - k(\phi A\phi - \phi^2 A) .$$

Thus $L_\xi Q = 0$ is equivalent to

$$[\phi A^2\phi - A\phi^2 A - k(\phi A\phi - \phi^2 A)]X = (\nabla_\xi Q)X . \tag{5.3}$$

Since Q is skew-symmetric, $\langle (\nabla_\xi Q)X, X\rangle = 0$ and so (5.3) implies that

$$\langle [\phi A^2\phi - A\phi^2 A - k(\phi A\phi - \phi^2 A)]X, X\rangle = 0 .$$

By putting $X = \xi$ in the above equation, we obtain

$$\langle \phi A\xi, \phi A\xi\rangle = -\langle A\phi^2 A\xi, \xi\rangle = 0 .$$

Thus, we have $A\xi = \alpha\xi$. From which, together with (2.2) and (2.4), (5.3) becomes

$$
\begin{aligned}
[\phi A^2\phi - A\phi^2 A - k(\phi A\phi - \phi^2 A)]X &= (\nabla_\xi \phi)AX + \phi(\nabla_\xi A)X + (\nabla_\xi A)\phi X \\
&\quad + A(\nabla_\xi \phi)X - k(\nabla_\xi \phi)X \\
&= \frac{\alpha}{2}(\phi^2 A - A\phi^2)X\,. \qquad (5.4)
\end{aligned}
$$

Now, we note that $L_\xi Q = 0$ is equivalent to "$A\xi = \alpha\xi$ and M satisfying (5.4)".

Next, let $X \in \Gamma(D)$ be a unit vector with $AX = \lambda X$ and let us restrict our discussion to the subset **G** at the moment. Then by using (2.5) and taking account of (5.4), we obtain

$$(\lambda - \overline{\lambda})(\lambda + \overline{\lambda} - k) = 0\,. \qquad (5.5)$$

This implies that λ is constant on M. Thus Theorem 2.1 and Theorem 2.2 assert that M is locally congruent to one of the model spaces. As an immediate consequence of Proposition 5.1, the conditions $L_\xi Q = 0$ hold for any real hypersurface of type A_0, A_1 and B. On the other hand, it follows form Theorem 2.3 that a real hypersurface of type A_2 satisfies (5.4).

Finally, let M be locally congruent to one of type C, D or E in CP^n. Set $t = \cot r$ ($0 < r < \frac{\pi}{4}$).

Then M has five distinct constant principal curvatures (cf. [11]):

$$\alpha = t - \frac{1}{t}\,, \quad \lambda_1 = t\,, \quad \lambda_2 = -\frac{1}{t}\,, \quad \lambda_3 = \frac{1+t}{1-t}\,, \quad \lambda_4 = \frac{t-1}{t+1}\,.$$

Let $X \in V_{\lambda_j}$ ($j = 1, 2$). Then we have $\phi AX = A\phi X$ or $\lambda_j = \overline{\lambda}_j$. Hence, λ_j is a solution of (5.5). Now, for $X \in V_{\lambda_j}$ ($j = 3, 4$), since $\phi V\lambda_3 = V_{\lambda_4}$ we have

$$\lambda_j + \overline{\lambda}_j = \lambda_3 + \lambda_4 = \frac{4t}{1-t^2} = -\frac{4}{\alpha}\,.$$

Therefore, λ_j is a solution of (5.5) when $k = -\frac{4}{\alpha}$. Accordingly, we conclude that M satisfies the condition $L_\xi Q = 0$.

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